Use of Galerkin's Method for Minimum-Weight Panels with Dynamic Constraints

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Galerkin's method is applied to the design of minimum-weight structures with dynamic constraints. The problems considered include the weight optimization of a simply supported beam and a panel, with the condition that their fundamental frequencies be the same as those of corresponding uniform-thickness structures. Galerkin's method also is used for the weight optimization of a semi-infinite panel and of a finite square panel, both of which have flutter speed constraints. Galerkin's technique is determined to be an effective method of finding approximate solutions to these structural optimization problems. The Galerkin solutions of the beam vibration and semi-infinite panel flutter problems compare favorably with exact numerical results. For the two-dimensional problems of panel vibration and flutter panel, initial rough estimates of the minimum-weight thickness distribution are calculated.

Nomenclature

D = panel flexural stiffness
D_1, D_2 = differential operators
E = Young's modulus
E_1, E_2 = error functions
I = area moment of inertia
L = length of beam or width of panel
M = Mach number
m = mass per unit length
n = frequency ratio
t = nondimensional thickness
w = nondimensional deflection
x = nondimensional length coordinate
y = nondimensional width coordinate
β = assumed function for thickness
θ = assumed function for deflection
λ_0 = flutter parameter
λ_i = Lagrange multipliers
μ = Poisson's ratio
ω = frequency
$(\ )_0$ = uniform thickness structure
$(\ )'$ = differentiation with respect to x
$(\ )_y$ = differentiation with respect to y

I. Introduction

THE field of weight minimization of structures with dynamic constraints has been studied vigorously for the past nine years. Having been concerned with optimum structures with specified strength requirements, investigators realized the potential of aircraft weight savings if optimization techniques were applied to structures constrained by a dynamic parameter, such as fundamental frequency, torsional divergence speed, bending-torsion flutter speed, or panel flutter speed. The goal of the subsequent studies has been to find the minimum-weight structure whose dynamic characteristics are the same as those of a given reference structure. These minimum-weight structures are constrained by stiffness requirements rather than strength requirements, which are the usual constraining factors in the preliminary design of aircraft structures today.

Historically, two distinctly different approaches have been taken to the dynamic optimization problem, one numerical and the other analytical. The numerical method employs the finite-element modeling of relatively complex structures. Turner[1] introduced this numerical approach by applying it to a weight optimization problem with a specified natural frequency. Many papers followed Turner's publication, employing more sophisticated methods of parameter optimization and applying these methods to complex aircraft structures involving several hundred degrees of freedom. The advantage of the numerical approach has been its ability to model easily very complicated structures. The disadvantage has been the large number of design variables involved. One is introduced for each element used in representing the structure. Thus for complex structures, an excessive number of design variables is handled by the parameter optimization procedure.

In contrast to the numerical method of structural optimization, the analytical method uses functional optimization procedures and is applied only to very simplified, and therefore less realistic, structures. McIntosh et al.[2] suggested that the study of these simplified structures using the calculus of variations would "make it possible to explore to the fullest the potential of aeroelastic optimization and to seek results of general applicability." The advantage of the analytical approach has been a reduction in the number of design variables in comparison with the numerical approach. Using the Galerkin procedure, the number of design variables is equal to the number of terms in the series representation of the function involved. Even for complicated structures, significantly fewer design variables are required.

The analytical approach to the weight optimization problem has been hampered by the lack of a means to solve the resulting differential equations. The numerical solutions are limited to one-dimensional structures and require a large amount of computer time. Also, closed-form solutions are probably nonexistent for panel problems. Galerkin's method appears to be one of the more promising techniques available. This paper will examine Galerkin's solution to the differential equations resulting from the weight minimization of some simple structure with dynamic constraints.

II. Problem Formulation

A. Optimization Problem

In general, the optimization problem can be stated as finding the minimum total-weight structure that has the same fundamental frequency (or the same flutter speed) as a given uniform-thickness structure. The cost functional in this case is the weight of the structure expressed as an integral of the thickness distribution in nondimensional form:

$$J = \int_0^1 \int_0^1 t \, dx dy \tag{1}$$

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The fourth-order differential equation describing the free transverse simple harmonic motion (partial differential equation) for the panel problem is written as a set of simultaneous first-order differential equations of the following form:

$$\Phi_i = f_i(\bar{w},t) - \frac{\partial w_i}{\partial x} \quad (i=1,2,...N) \tag{2}$$

where N is the number of first-order differential equations necessary to represent the Euler equation. The first-order differential equations impose a set of N state variable constraints on the weight minimization problem. The N constraints are appended to the total weight cost functional by the Lagrange multiplier method, and a Bolza-type problem is formed:

$$F = t + \sum_{i=1}^{N} \lambda_i \Phi_i \tag{3}$$

The necessary conditions for the minimization of the function F are described by the following Euler-Lagrange equations:

$$\frac{\partial F}{\partial w} - \frac{\partial}{\partial x}\left(\frac{\partial F}{\partial w_x}\right) - \frac{\partial}{\partial y}\left(\frac{\partial F}{\partial w_y}\right) = 0 \tag{4}$$

$$\partial F / \partial t = 0 \tag{5}$$

where $w_x = \partial w/\partial x$, $w_y = \partial w/\partial y$, and w represents each of the variables w_i.

Equations (2-4) and (5) form a set of $2N+1$ equations for the N state variables (w_i), N Lagrange multipliers (λ_i), and one control variable (t). The boundary conditions are supplied by the geometric and natural constraints on the structural member and by the transversality conditions.

In each problem encountered, a relationship between the state variables (w_i) and the Lagrange multipliers (λ_i) is found to eliminate the multipliers from the problem. Only two differential equations then remain to be solved simultaneously, the Euler equation and the optimality condition (5). These two equations are coupled by the deflection variable (w) and the thickness variable (t).

B. Galerkin's Method

The Galerkin technique is one of a group of weighted residual methods employed in solving differential equations encountered in mechanics. A series solution (polynomial, trigonometric, etc.) is assumed for the unknown variables in the differential equations. For the weight optimization problems investigated here, the unknowns are the thickness function (t) and the deflection function (w). Series solutions of the following form are employed:

$$w(x) = \sum_{i=1}^{N} a_i \theta_i(x) \tag{6}$$

$$t(x) = \sum_{i=1}^{M} b_i \beta_i(x) \tag{7}$$

where a_i and b_i are unknown coefficients, and θ_i and β_i are assumed functions such as $\sin \pi x$. These series must satisfy the geometric and boundary conditions of the problem exactly. Equations (6) and (7) are introduced into the Euler equation and optimality condition, and two error functions, E_1 and E_2, are produced:

$$D_1\left[\sum_i a_i \theta_i(x), \quad \sum_i b_i \beta_i(x)\right] = E_1(x) \tag{8}$$

$$D_2\left[\sum_i a_i \theta_i(x), \quad \sum_i b_i \beta_i(x)\right] = E_2(x) \tag{9}$$

where D_1 is the differential operator for the Euler equation, and D_2 is the differential operator for the optimality condition.

The goal of the Galerkin procedure is to determine systematically the unknown coefficients, a_i and b_i, which minimize the error functions E_1 and E_2. This goal is accomplished by imposing the following orthogonality conditions on the error functions:

$$\int_0^1 E_1 \theta_j \mathrm{d}x = 0 \quad (j=1,2,...N) \tag{10}$$

$$\int_0^1 E_2 \beta_j \mathrm{d}x = 0 \quad (j=1,2,...N) \tag{11}$$

In effect, Galerkin's method makes the error orthogonal to each assumed function, θ_j or β_j. The result is a set of $N+M$ nonlinear algebraic equations in terms of the unknown coefficients, a_i and b_j. A Newton-Raphson scheme is used to solve the algebraic equations.

III. Problems and Results

A. Beam Vibration

The first structure to be analyzed is a simply supported beam of constant width and length. The thickness of the beam is varied to find the minimum-weight beam that has the same fundamental frequency as a uniform-thickness beam. The Euler equation describing free transverse vibration of the beam in nondimensional form is

$$(t^3 w'')'' - \omega^2 (m_0 L^4 / EI_0) t w = 0 \tag{12}$$

The design requirement imposed on the optimum beam is that it have the same fundamental frequency as a uniform-thickness beam. This constraint is introduced into the formulation by first noting that the square of the fundamental frequency for a uniform-thickness beam is

$$\omega^2 - \pi^4 (EI_0 / m_0 L^4) \tag{13}$$

and then substituting Eq. (13) into Eq. (12):

$$[t^3(x) w''(x)]'' - \pi^4 t(x) w(x) = 0 \tag{14}$$

The fourth-order Euler equation is broken up into four first-order differential equations, appended to the cost function and put into the Bolza form [Eq. (3)]. The Euler-Lagrange necessary conditions are combined into one fourth-order differential equation involving λ_4

$$(t^2 \lambda_4'')'' - \pi^4 t \lambda_4 = 0 \tag{15}$$

As Eq. (15) is the same as Eq. (14) with λ_4 replacing w and the boundary conditions are equivalent, λ_4 is a constant multiple of w. Letting A be the constant multiple, the relationship can be written as

$$\lambda_4 = A w \tag{16}$$

Using Eq. (16). the optimality condition, Eq. (5), becomes

$$t^2 = [1 + \pi^4 (Aw)] / 3A (w'')^2 \tag{17}$$

Equation (17) indicates that the constant multiple A must be positive for a positive thickness to be computed. For convenience, A is set equal to 1, and Eq. (17) becomes

$$3t^2 (w'')^2 - (1 + \pi^4 w^2) = 0 \tag{18}$$

The problem now is reduced to the solution of differential equations (14) and (18), which are coupled by the thickness (t) and deflection (w) functions.

Galerkin's technique for solving Eqs. (14) and (18) requires choosing series for t and for w which satisfy the boundary conditions. A quasilinearization numerical solution[3] of this problem has shown that the deflection shape (w) and thickness distribution (t) are symmetric about the line $x = ½$. Therefore, an odd sine series is used for both w and t as it satisfies the boundary conditions and is symmetric:

$$w = \sum_{i=\text{odd}}^{N} a_i \sin i\pi x \tag{19}$$

$$t = \sum_{i=\text{odd}}^{M} b_i \sin i\pi x \tag{20}$$

Initially, a one-term sine series for both thickness and deflection was assumed. The solution for the coefficients was calculated easily by hand to be $a_1 = 0.0877$ and $b_1 = 1.118$. The mass ratio, the ratio of the total weight of the calculated optimum structure to the weight of the uniform structure, was 0.712. The thickness distribution for the one-term sine series solution is shown in Fig. 1, together with the numerical solution. The deflection is pictured in Fig. 2 with its numerical counterpart. It appears from Fig. 2 that no more than a one-term sine series is required to represent adequately the deflection shape. However, Fig. 1 indicates that more terms in the series will be necessary for the thickness distribution.

The one-term sine series originally was intended to suffice for deflection, with more terms being added for thickness; however, all attempts to use this procedure resulted in physically unacceptable answers. The procedure then was altered to allow for two terms in the deflection series and two in the thickness series; this was successful. Although the coefficient of the second term in the deflection series (a_3) was very small, it was quite important in the calculation of the first sine term for the thickness (b_1). In fact, neglecting the a_3 terms tends to increase dramatically the value of b_1. The reason for the importance of a_3 is that the associated $\sin 3\pi x$ terms are differentiated twice when introduced into the optimality condition, resulting in an immediate multiplication by nine. The resulting term in the algebraic equations is very significant in the computation of the coefficients for the thickness series. Another term in the deflection series $a_5 \sin 5\pi x$ was added to determine its effect on the solution.

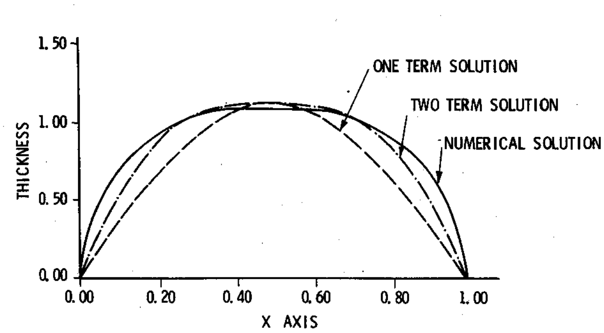

Fig. 1 Thickness distribution of the optimum simply supported beam with constrained fundamental frequency.

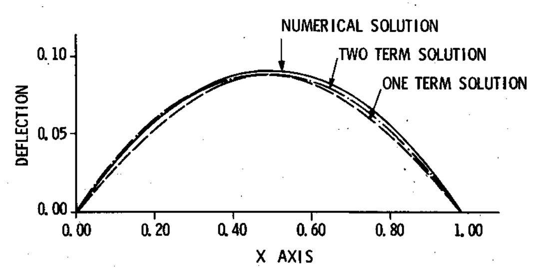

Fig. 2 Optimum simply supported beam deflection with constrained fundamental frequency.

The changes to the thickness and deflection were insignificant.

A procedure for using Galerkin's technique with these optimization problems was formulated as a result of experience with the beam vibration problem. As a general rule, after selecting the series to represent the thickness distribution, terms will be added to the deflection series until only minimal changes are obtained by the addition of another deflection term. This rule complicates the calculations greatly but is quite necessary to obtain useful answers.

The results for the two-term sine series for deflection and thickness are

$$t(x) = 1.235 \sin \pi x + 0.146 \sin 3\pi x \tag{21}$$

$$w(x) = 0.09088 \sin \pi x + 0.0036 \sin 3\pi x \tag{22}$$

These equations are illustrated in Figs. 1 and 2, along with their corresponding numerical solutions.

B. Panel Vibration

Using Galerkin's technique, the analysis of the optimum beam can be extended easily to the two-dimensional problem of finding the minimum-weight square panel that has the same fundamental frequency as a uniform panel. The panel is simply supported on all four edges. The nondimensional equation of motion for vibration of a flat plate with fixed fundamental frequency is

$$\nabla^2(t\nabla^2 w) - (1-\mu)\left[(t^3 w_{xx})_{yy} + (t^3 w_{yy})_{xx} - 2(t^3 w_{xy})_{xy}\right] - 4\pi^4 t w = 0 \tag{23}$$

The formulation proceeds exactly along the same lines as the optimum beam problem. Equation (23) is resolved into 11 first-order equations. These state constraint equations are appended to the cost functional in the Bolza form. As in the beam problem, the 11 Euler-Lagrange equations can be combined into one fourth-order differential equation involving λ_{11}:

$$\nabla^2[t^3\nabla^2\lambda_{11}] - (1-\mu)\{[t^3(\lambda_{11})_{xx}]_{yy} + [t^3(\lambda_{11})_{yy}]_{xx} - 2[t^3(\lambda_{11})_{xy}]_{xy}\} - 4\pi^4 t\lambda_{11} = 0 \tag{24}$$

Equations (23) and (24) are similar equations with similar boundary conditions. Therefore, λ_{11} is a constant multiple of w., and this multiple may be set equal to one. Using $\lambda_{11} = w$, the optimality condition becomes

$$3t^2[(\nabla^2 w)^2 - 2(1-\mu)(w_{xx}w_{yy} - w_{xy}^2)] - (1 + 4\pi^4 w) = 0 \tag{25}$$

The optimum panel problem now is reduced to solving the two coupled partial differential equations (23) and (25).

Unlike other solution techniques, which are limited to one-dimensional problems, Galerkin's method easily can be extended to two-dimensional situations. It is known that the mode shape of the uniform panel vibrating at its fundamental

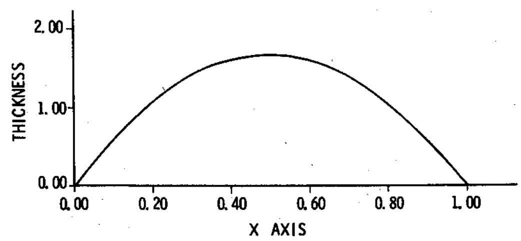

Fig. 3 Thickness distribution of the optimum square panel with constrained fundamental frequency.

frequency is symmetric about both the x and y midpoints. A similar solution is assumed for the mode shape of the optimum panel. Therefore, the odd sine series in x and y is used for the deflection shape:

$$w=\sum_{i=\text{odd}}^{N}\sum_{j=\text{odd}}^{N} a_{ij}\sin i\pi x \sin j\pi y \tag{26}$$

A series similar to the deflection shape is chosen for thickness:

$$t=\sum_{i=\text{odd}}^{N}\sum_{j=\text{odd}}^{M} b_{ij}\sin i\pi x \sin j\pi y \tag{27}$$

To obtain just a first estimate of the solution, two terms were assumed. The thickness distribution for the minimum-weight square panel with constrained natural frequency was calculated to be

$$t=(1.717\sin\pi x+0.0643\sin 3\pi x)\sin\pi y \tag{28}$$

This equation is illustrated in Fig. 3 by showing the thickness distribution at the midplane of the panel. The mass ratio for this rough estimate of the minimum-weight panel is 0.71.

C. Semi-Infinite Panel Flutter

Determining the minimum-weight panel with constrained flutter speed is usually more difficult than finding the optimum panel with a constant fundamental frequency. However, using Galerkin's procedure, the flutter problem simply adds a few terms to the final algebraic equations. The semi-infinite panel is studied in order to simplify the problem by putting it into one-dimensional form. The panel has finite length but infinite width and is essentially undergoing cylindrical bending. Thickness and mode shape are assumed to be independent of the y variable. Therefore, the semi-infinite panel problem can be viewed as a beam of unit width.

The one-dimensional panel is assumed to be initially flat and to have no in-plane stresses at the midplane. The panel is simply supported at the front and back edges and is of solid construction. It is subjected to a high supersonic airflow in the positive x direction on one side of the panel. The Mach number is sufficiently high so that the piston theory may be used to represent the aerodynamic forces ($M\approx 1.6$). Assuming also that structural damping is negligible, the Euler equation[4] for the semiinfinite panel problem in nondimensional form is

$$(t^3w'')''+\lambda_0 w'-(n\pi)^4 tw=0 \tag{29}$$

where λ_0 is the flutter parameter and $n^2=\omega^2/\omega_0^2$ is the ratio of the flutter frequency (ω) to the fundamental frequency of a semi-infinite panel in vacuum (ω_0).

The formulation of the optimization equations proceeds along the same lines as the beam vibration problem. As in the two previous problems, the Euler-Lagrange equations can be combined into one fourth-order differential equation:

$$(t^3\lambda_4)''-\lambda_0\lambda_4'-kt\lambda_4=0 \tag{30}$$

where $k=(n\pi)^4$. In the constrained vibration cases, it was found that the equation of motion and the Euler-Lagrange equation were exactly the same in the two variables w and λ_4. However, in the constrained flutter problem, a comparison of Eqs. (29) and (30) shows that the flutter term has reversed its sign in the Euler-Lagrange equation. Using a suggestion attributed to Turner,[1] Armand and Vitte[5] found a relationship between the Lagrange multipler and the deflection:

$$\lambda_4=B\bar{w} \tag{31}$$

where $\bar{w}(x)=w(1-x)$ and B is a constant multiple. Using Eq. (31), the optimality condition becomes

$$t^2=(1+kBw\bar{w})/3Bw''\bar{w}'' \tag{32}$$

Weisshaar[3] made an extensive analysis of the sign of B and found that meaningful answers could be obtained only for $B<0$. For convenience, B is set equal to -1, and Eq. (32) is written as

$$3t^2w''\bar{w}''+1-kw\bar{w}=0 \tag{33}$$

As indicated in Weisshaar's report,[3] the numerical solution to Eq. (29) and (33) is rather difficult because four additional first-order equations must be introduced for the deflection terms $\bar{w}=w(1-x)$. However, this is not the situation using Galerkin's technique. The following series is assumed for w:

$$w=\sum_{i=1}^{N} a_i\sin i\pi x \tag{34}$$

A symmetric thickness distribution has been assumed for this solution and is represented by an odd sine series:

$$t=\sum_{i=\text{odd}}^{M} b_i\sin i\pi x \tag{35}$$

Using two terms in the thickness series, the results for semi-infinite panel flutter were

$$t=1.08\sin\pi x+0.535\sin 3\pi x \tag{36}$$

Equation (36) is illustrated in Fig. 4, together with the corresponding exact numerical solution. In Fig. 4, the Galerkin two-term series solution predicts quite well the same general shape of the panel as the exact numerical solution. The mass ration for the Galerkin approximation is 0.81 as compared to 0.87. To improve the thickness distribution approximation would require a combination of a number of the higher odd sine terms.

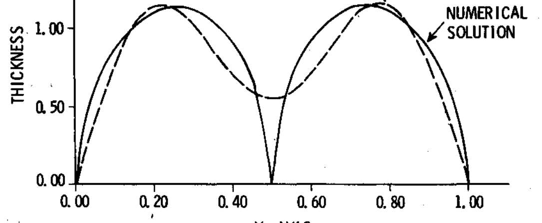

Fig. 4 Thickness distribution of the optimum semi-infinite panel with fixed flutter parameter.

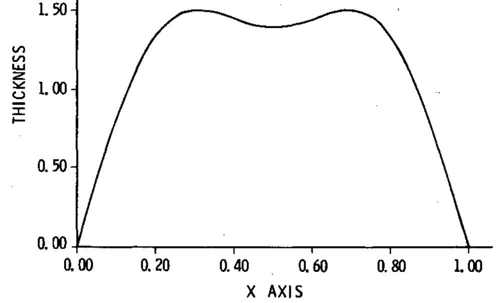

Fig. 5 Thickness distribution of the optimum square panel with fixed flutter parameter.

D. Panel Flutter

To previous investigators, such as Armand and Vitte,[5] the prospect of solving an aeroelastic optimization problem involving a finite-span panel seemed to be nearly impossible. However, with the aid of Galerkin's method and with the experience gained in solving some simplified problems, a solution appears to be attainable.

In establishing the necessary equations to determine the minimum-weight square panel that has a flutter speed equal to that of a uniform panel, the same assumptions are used as in the semi-infinite panel flutter case, with a few exceptions. The y dimension of the panel is finite, and all quantities have a dependence on the y variable. Also, the panel is simply supported on all four edges.

The Euler equation for the flutter of a square, flat panel in nondimensional form is

$$\nabla^2(t^3\nabla^2 w) - (1-\mu)\left[(t^3 w_{xx})_{yy} + (t^3 w_{yy})_{xx} - 2(t^3 w_{xy})_{xy}\right] + \lambda_0 w_x - 4ktw = 0 \tag{37}$$

As in the semi-infinite panel problem, a symmetric thickness distribution about the panel midpoint is assumed to obtain

$$\lambda_{11} = B\bar{w} \tag{38}$$

where $\bar{w} = w(1-x,\ 1-y)$. The modal constant B will be set equal to -1. Using Eq. (38), the optimality criteria becomes

$$3t^2\left[\nabla^2 w\nabla^2\bar{w} - (1-\mu)(w_{xx}\bar{w}_{yy} + w_{yy}\bar{w}_{xx} - 2w_{xy}\bar{w}_{xy})\right] + 1 - 4kw\bar{w} = 0 \tag{39}$$

Equations (37) and (39) are the two coupled partial differential equations to be solved by Galerkin's method. As the series solution for the thickness must be symmetric about the midlength in both the x and y directions, the odd sine series is chosen:

$$t(x,y) = \sum_{i=\text{odd}}^{N}\sum_{j=\text{odd}}^{M} b_{ij}\sin i\pi x \sin j\pi y \tag{40}$$

However, all of the sine terms must be included in the series solution for the deflection w:

$$w(x,y) = \sum_{i=1}^{N}\sum_{j=1}^{M} a_{ij}\sin i\pi x \sin j\pi y \tag{41}$$

The solution to the problem of finding the least-weight square panel with constrained flutter speed became quite routine after having solved the three previous structural problems. As in the panel vibration case, just an initial rough estimate of the solution was determined. The thickness and deflection in the y direction were allowed to vary only as $\sin\pi y$. The thickness series contained the first two symmetric functions in x, and the deflection series included the first two symmetric and first antisymmetric terms. The solution for the optimum square panel flutter case was

$$t = (1.73\sin\pi x + 0.34\sin 3\pi x)\sin\pi y \tag{42}$$

This equation is illustrated in Fig. 5. The figure shows the thickness at the midplane of the panel in the y direction where $\sin\pi y = 1$. A comparison of the square panel solution with the semi-infinite panel answer shows a large increase of the thickness near the quarter and three-quarters points. This increase tends to compensate for the general decrease of thickness on the remaining sections of the panel.

IV. Conclusions

Galerkin's technique is an effective method of finding approximate solutions to structural optimization problems described by a set of nonlinear differential equations. The major advantage of this technique is its applicability to the two-dimensional problems of panel vibration and panel flutter for which no numerical technique of solution is available except for the finite-difference method.[6] In the case of one-dimensional problems, if an exact numerical answer is desired, the Galerkin solution can save considerably on the total computer time by providing good guesses for the initial unknown variables. For any structural optimization problem, this method has the additional advantage of requiring only a minimum amount of computer time as compared to the numerical methods and reducing the number of design variables used in the optimization procedure.

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